<u>Derivation of orbital dynamics scalar equation and attitude dynamics equation</u>.

The paper is mainly focus on the corresponding relationship between the thrust vector and the attitude of the electric sail. For simplicity, the flexibility of tethers is ignored in this paper.

<u>Reference Frame</u>

Firstly, three frames are introduced which are:

- Body Frame - $o_b x_b y_b z_b$
- Orbital Frame - $o_o x_o y_o z_o$
- Heliocentric elliptical inertial frame - $o_i x_i y_i z_i$

The origin of the body frame is at centre of mass of the sail. (Fig. 2)

- x_b axis is in the direction of the given reference tether.
- z_b axis is along the normal of the sail.
- y_b axis forms a right-handed system.

The origin of the orbital frame is at centre of mass of the sail. (Fig. 3)

- z_o axis is along sun-spacecraft direction.
- y_o axis is perpendicular to normal of elliptic plane and z_o axis.
- x_o axis forms a right-handed system.

The origin of the inertial frame is centre of mass of the sun. (Fig. 3)

- x_i axis is in direction of sun equinox.
- z_i axis is along the normal of the elliptical plane.
- y_i axis forms a right-handed system.

The attitude of the sail in the orbital frame can be described by three angles ϕ, θ and ψ, and the rotation sequence from the orbital frame to the body frame is x(ϕ) ->y(θ)->z(ψ). Based on the basic knowledge of matrix transformation, the transition matrix from $o_o x_o y_o z_o$ to $o_b x_b y_b z_b$ is

$$\boxed{\boldsymbol{A}_{bo}(\phi,\theta,\psi)=\boldsymbol{R}_z(\psi)\boldsymbol{R}_y(\theta)\boldsymbol{R}_x(\phi) \qquad (1.)}$$

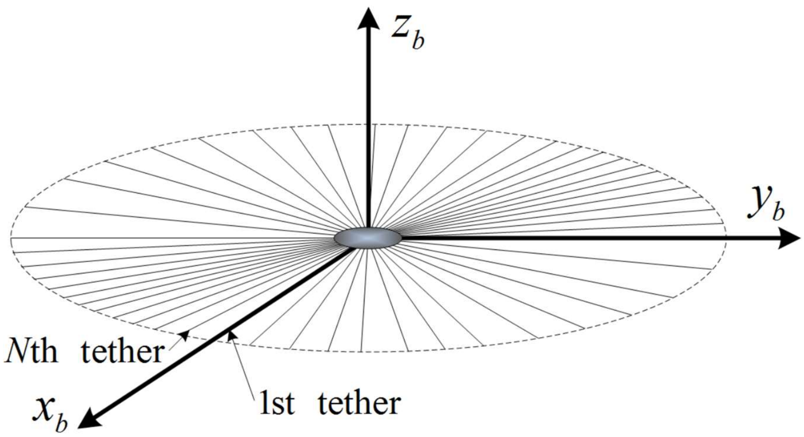


Fig 2. Body reference frame.
Note – Total number of tethers is N and tethers are numbered counterclockwise.

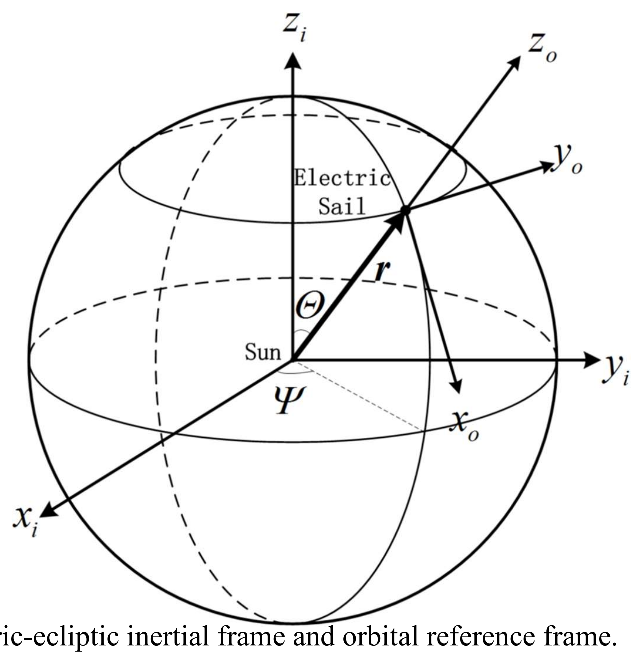


Fig 3. Heliocentric-ecliptic inertial frame and orbital reference frame.

Where A_{bo} in Eq. is,

$$A_{bo} = \begin{bmatrix} cos\psi & sin\psi & 0 \\ -sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{bmatrix} \begin{bmatrix} cos\phi & sin\phi & 0 \\ -sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_{bo} = \begin{bmatrix} cos\theta cos\psi & sin\phi sin\theta + cos\phi sin\psi & -cos\phi sin\theta cos\psi + sin\theta sin\psi \\ -cos\phi cos\psi & -sin\phi sin\theta sin\psi + cos\phi cos\psi & -cos\phi sin\phi sin\psi + sin\phi cos\psi \\ sin\theta & -sin\theta cos\theta & cos\phi cos\theta \end{bmatrix}$$

Thrust Vector

Using Reference paper [9]

Propulsive thrust per unit length of tether is:

$$\boxed{\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}l} = 0.18\max(0, V_0 - V_1)\sqrt{\varepsilon_0 m_p n_w}\mathbf{u}_\perp \qquad (2.)}$$

Where,
V_o – tether voltage.
V_i – electric potential corresponding to the kinetic energy of the solar wind ions.
n_w – solar wind number density.
$u_\perp$ – component of solar wind perpendicular to charged tether

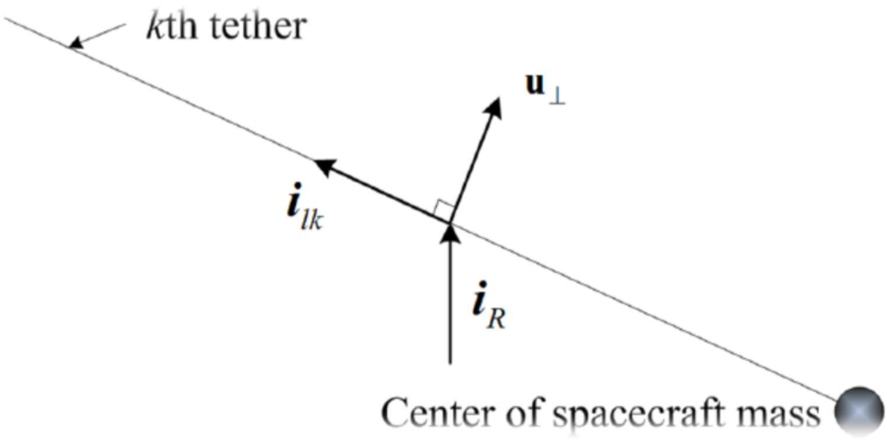


Fig 4. Thrust vector of kth tether.

Components of i_R
Perpendicular component – $i_\perp$
Parallel component - i_l

From Fig.4,

$$u\hat{R} = ucos\theta\hat{\imath}_l + usin\theta\widehat{\imath_\perp}$$

$here\ \theta\ is\ angle\ between\ \hat{R}\ and\ \hat{\imath}$

$$u_\perp = usin\theta\ \widehat{\imath_\perp}$$

$$let\ new\ direction\ be\ sin\theta\widehat{\imath_d} = \widehat{\imath_l} * \widehat{\imath_R}$$

$$\widehat{\imath_d} = \widehat{\imath_l} * \widehat{\imath_\perp}$$

We need $\widehat{\imath_\perp}$

So,

$$\widehat{\imath_\perp} = \widehat{\imath_d} * \widehat{\imath_l}$$

$$\widehat{\imath_\perp} = \widehat{\imath_l} * \frac{\widehat{\imath_R} * \widehat{\imath_l}}{sin\theta}$$

Hence

$$u_{\perp} = usin\theta \left[\widehat{\iota_l} * \frac{\widehat{\iota_R} * \widehat{\iota_l}}{sin\theta} \right]$$

$$\boxed{\mathbf{u}_{\perp} = u\mathbf{i}_l \times \mathbf{i}_R \times \mathbf{i}_l \qquad (3.)}$$

where,

u – magnitude of the solar wind

i_l – unit direction vector of tether

i_R – unit vector of sun- spacecraft reference direction

From Reference paper [11]

Because of the existence of centrifugally stabilizing auxiliary tethers, it can be assumed that electric sail tethers always distributed evenly in the $o_b x_b y_b z_b$ plane. Therefore, the unit direction vector of the kth tether in the body frame is $[\cos(2\pi(k-1)/N)\ \sin(2\pi(k-1)/N)\ 0]^T$. Consequently, the unit direction vector of the kth tether in the orbital frame can be given by

$$\boxed{\boldsymbol{i}_{lk} = \boldsymbol{A}_{ob} \begin{bmatrix} \cos(2\pi(k-1)/N) \\ \sin(2\pi(k-1)/N) \\ 0 \end{bmatrix} \qquad (4.)}$$

Now substituting Eq (3) into Eq (2), the thrust vector per unit length can be obtained as:

$$\boxed{\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}l} = \sigma \boldsymbol{i}_l \times \boldsymbol{i}_R \times \boldsymbol{i}_l \qquad (5.)}$$

Where,

$$\sigma = 0.18 \max(0, V_0 - V_1) \sqrt{\varepsilon_0 m_p n_w u^2}$$

σ = magnitude of thrust per unit length, when solar wind direction is perpendicular to the tether.

From Reference paper [6]

Electric sail thrust decays as $\left[\frac{1}{r}\right]$

The thrust per unit length can be written as:

$$\frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}l} = \sigma_{\oplus}\frac{r_{\oplus}}{r}\boldsymbol{i}_l \times \boldsymbol{i}_R \times \boldsymbol{i}_l \qquad (6.)$$

Where,

$\sigma_{\oplus}$ = magnitude of the force per unit length when sun-spacecraft distance is $r_{\oplus}$ = 1au. And the solar wind is perpendicular to tether.

Now, Integrating Eq. 6

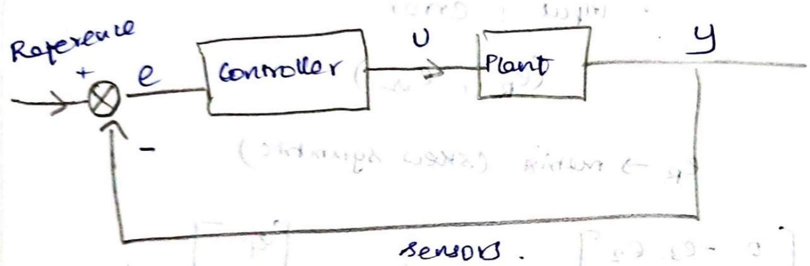
The thrust vector of kth tether can be obtained as

$$\boldsymbol{F}_k = l\sigma_{\oplus k}\left(\frac{r_{\oplus}}{r}\right)\boldsymbol{i}_l \times \boldsymbol{i}_R \times \boldsymbol{i}_l \qquad (7.)$$

Summing over the thrust vector of single tethers, we can obtain the total thrust of the electric sail:

$$\boldsymbol{F}(\phi, \theta, \psi, r, l, \sigma_{\oplus 1}, \ldots, \sigma_{\oplus N}) = \sum_{k=1}^{N} \boldsymbol{F}_k(\phi, \theta, \psi, r, l, \sigma_{\oplus k}) \qquad (8.)$$

Attitude tracking system:



There are two main components that run in loop

(i) Controller

(ii) Plant

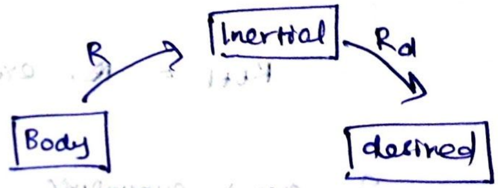
The Controller and the plant are the two important things of this feedback loop,
The feedback loop runs these two processes repeatedly to track the system (reference).
The state of our system is

$$(X)\ \text{State} = \begin{bmatrix} R \\ \Omega \end{bmatrix} \begin{matrix} \text{rotation matrix} \\ \text{angular velocity} \end{matrix}$$

As our system is a full feedback system. The error is calculated and sent to the controller as follows error e

$$e_{\Omega_t} = \Omega_{desired} - \Omega_t$$

$$e_R = \frac{R_d^T R - R^T R_d}{2}$$



skew symmetric matrix

The error calculated from the sensor feedback and the reference system is fed into the controller to make the system track. We are using the simple PD controller in this system.

Controller

- input : error
  (e_R, e_Ω)

$e_R \to$ matrix (skew symmetric)

$$e_R = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix} \xrightarrow[\text{(Veemap)}]{} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \cdot e_{R\_V}$$

vector form.

$$U = -K_p e_{R\_V} - K_D e_\Omega$$

- output : u (matrix)

As the error from the rotation matrix cannot be directly used. It is used as vee-map transfer (skew symmetric matrix to vector). After having a vector of the states PD controller is applied. The moment is given as the output from the controller

Plant is the real system, which takes the moment as the input from the controller, the rotational dynamics governs the plant. Based on the input (M) our new states are obtained from the plant as shown in the below figure

<u>Plant</u>

input U output X_{t+1}

dynamics

$$\dot{\omega} = J^{-1}\left(-\hat{\omega} \times (J \times \bar{\omega}) + U\right)$$

$$\omega_{t+1} = \omega_t + \dot{\omega}\, dt$$

$$R_{t+1} = \cancel{R}, \exp(\hat{\omega}\, dt) \times R_t$$

$\hat{\omega}$ skew symmetric matrix of Ω (angular velocity)

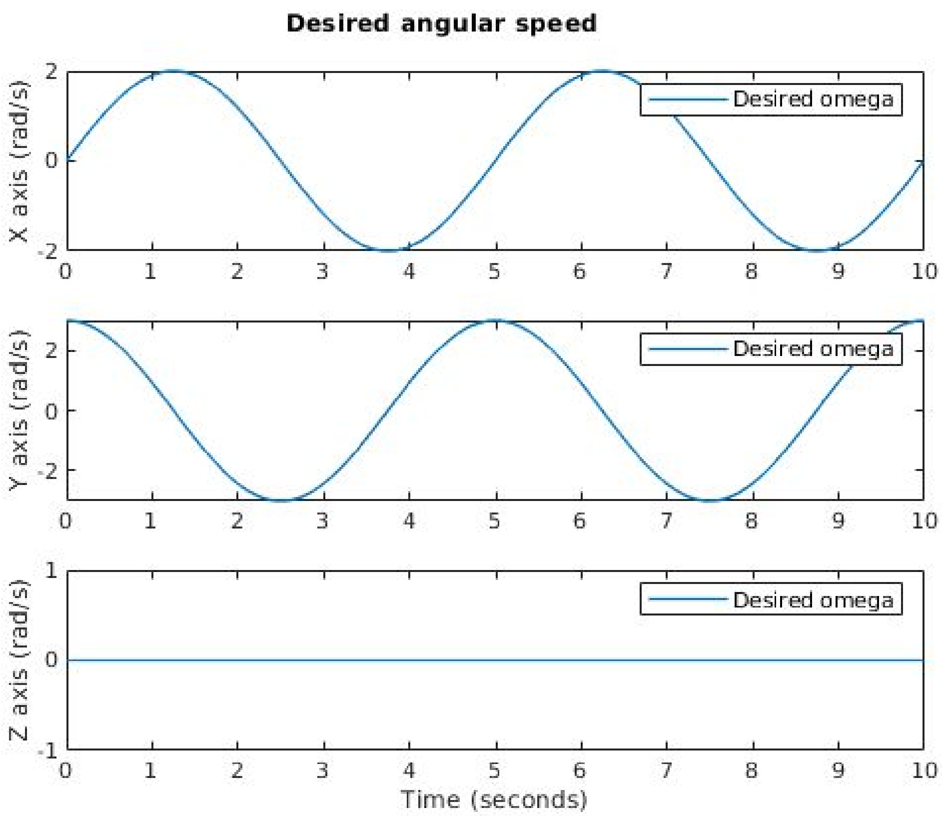
By Rodrigues formula $\exp(\hat{\omega}\, dt) = I + \sin\theta\, \hat{\omega} + [\hat{\omega}]^2 (1-\cos\theta)$

$$\theta = |\omega|\, dt$$

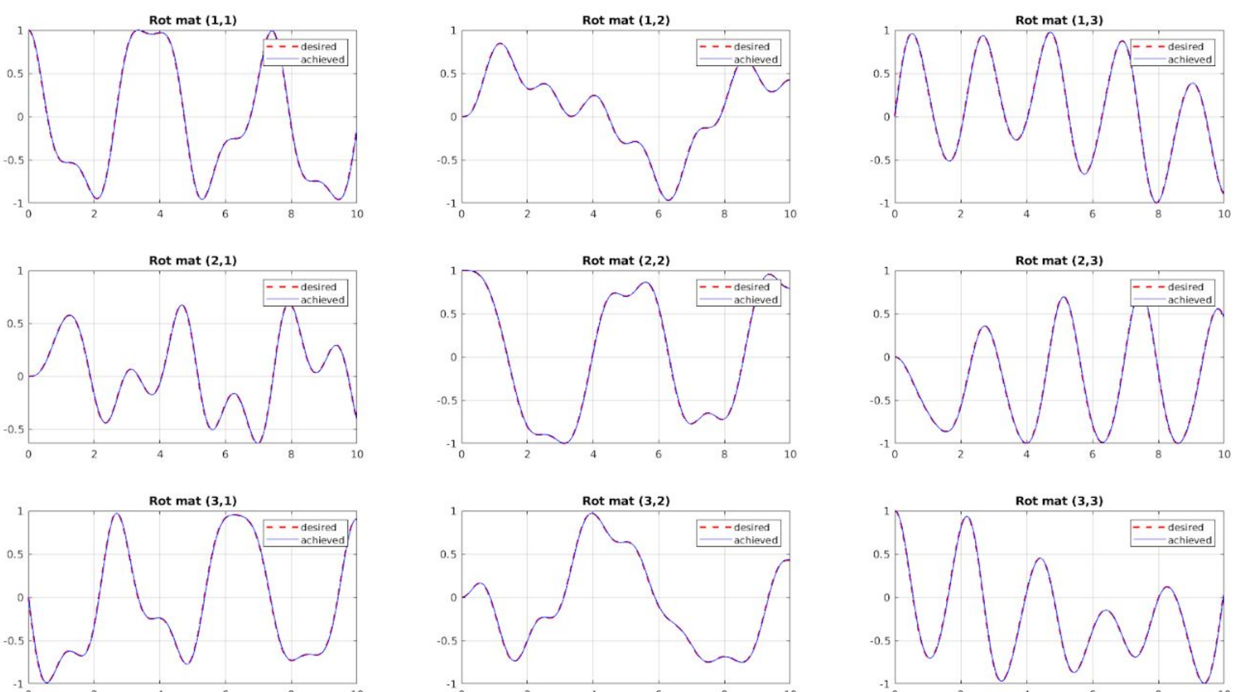
The main loop runs the Controller and the Plant continuously to track the system. The tuning of the system should be done to make the system robust.

Gain kp = 39.1;
Gain kd = 1035.15255;

The attitude tracking algorithm was coded in the matlab. The reference system was taken as the



Based on this reference angular velocity, the desired rotation matrix was developed. The attitude tracking algorithm was then computer with initial gains. The results obtained are below



The 9 states of the rotation matrix are plotted, red is the desired state and blue is the tracked state. Further tuning will be done to make the system accurate.

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